THE EULERIAN NUMBERS ON RESTRICTED CENTROSYMMETRIC PERMUTATIONS

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Abstract. We study the descent distribution over the set of centrosymmetric permutations that avoid the pattern of length 3. Our main tool in the most puzzling case, namely, $\tau = 123$ and n even, is a bijection that associates a Dyck prefix of length $2n$ to every centrosymmetric permutation in S_{2n} that avoids 123.

Keywords: restricted permutations, centrosymmetric permutations, Dyck prefixes, Eulerian numbers.

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1. Introduction

A permutation $\sigma \in S_n$ is *centrosymmetric* if $\sigma(i)+\sigma(n+1-i) = n+1$ for every $i = 1, \ldots, n$. Equivalently, σ is centrosymmetric whenever $\sigma^{rev} = \sigma^c$, where rev and c are the usual reverse and complement operations. The subset C_n of centrosymmetric permutations is indeed a subgroup of S_n that, in the even case, is isomorphic to the hyperoctahedral group $B_{\frac{n}{2}}$, the natural B-analogue of the symmetric group.

Centrosymmetric permutations have been extensively studied in recent years from different points of view. For example, the present authors [1] studied the descent distribution (or Eulerian distribution) over the subset of centrosymmetric involutions, while Guibert and Pergola [5] and Egge [3] studied some properties of C_n from the pattern avoidance perspective.

In this paper we merge these two points of view, and analyze the descent distribution over the set $C_n(\tau)$ of centrosymmetric permutations that avoid a given pattern $\tau \in S_3$.

As well known, the six patterns in S_3 are related as follows:

- $321 = 123^{rev}$,
- $231 = 132^{rev}$,
- $213 = 132^c$,
- $312 = (132^c)^{rev}$.

Since a permutation σ is centrosymmetric whenever σ^{rev} and σ^c are centrosymmetric, in order to determine the distribution of the descent statistic over $C_n(\tau)$, for every $\tau \in S_3$, it is sufficient to examine the distribution of descents over the two sets $C_n(132)$ and $C_n(123)$.

In both cases, our starting point is the characterization of the elements in $C_n(\tau)$, already appearing in [3]. In the case $\tau = 132$, this characterization allows us to easily determine the descent distribution.

The case $\tau = 123$ presents some more challenging aspects. First of all, we observe that the sets $C_{2k}(123)$ and $C_{2k+1}(123)$ have substantially different features. In fact, the set $C_{2k+1}(123)$ is in bijection with the set $S_k(123)$ of 123-avoiding permutations. In this case, the descent distribution over C_{2k+1} can be trivially deduced from the descent distribution over $S_k(123)$, appearing in [2].

In the even case, we define a bijection Φ between the set of centrosymmetric permutations in $C_{2n}(123)$ and the set of Dyck prefixes of length $2n$. The map Φ yields a bijective proof of the result $|C_{2n}(123)| = \binom{2n}{n}$, that have been proved in [3] with enumerative techniques.
Moreover, the bijection Φ reveals to be a powerful tool in determining the descent distribution over $C_{2n}(123)$. In fact, the Dyck prefix $\Phi(\sigma)$ can be split according to its last return decomposition into subpaths that are either Dyck paths or elevated Dyck prefixes, namely, Dyck prefixes with no intersections with the x-axis, apart from the origin. The study of the descent distribution over the sets of permutations that correspond to Dyck prefixes of these two kinds leads to an explicit expression of the bivariate generating function

$$T(x,y) = \sum_{n\geq 0} \sum_{\sigma \in C_{2n}(123)} x^n y^{\mathrm{des}(\sigma)}.$$

2. Preliminaries

2.1. **Permutations.** Let $\sigma \in S_n$ and $\tau \in S_k$, $k \leq n$, be two permutations. We say that σ *contains* the pattern τ if there exists a subsequence $\sigma(i_1)\,\sigma(i_2), \ldots \sigma(i_k)$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, that is order-isomorphic to τ. We say that σ *avoids* τ if σ does not contain τ. Denote by $S_n(\tau)$ (respectively $C_n(\tau)$) the set of τ-avoiding permutations in S_n (resp. C_n), where C_n denotes the set of centrosymmetric permutations in S_n.

We recall that, given a permutation $\sigma \in S_n$, one can partition the set $\{1, 2, \ldots, n\}$ into intervals $I_1, \ldots, I_t$, with $I_j = \{k_j, k_j+1, \ldots, k_j+h_j\}$, $h_j \geq 0$, such that $\sigma(I_j) = I_j$ for every j. The restrictions of σ to the intervals in the finest of these decompositions are called the *connected components* of σ. A permutation σ with a single connected component is called *connected*. A permutation is called *right connected* if σ^{rev} is connected. The notion of right connected component of a permutation is defined in the obvious way.
For example, the permutation

$$\rho = 2\ 7\ 6\ 1\ 3\ 5\ 4$$

is right connected, while

$$\sigma = 5\ 7\ 6\ 4\ 2\ 1\ 3$$

is not.

Note that the right connected components of a centrosymmetric permutation are mirror symmetric.
In the following example, the centrosymmetric permutation τ is split into its connected components:

$$\tau = 7\ 8\ |6\ |4\ 5\ |3\ |1\ 2.$$

We say that a permutation σ has a *descent* at position i if $\sigma(i) > \sigma(i+1)$. The set of descents of σ is denoted by $\mathrm{Des}(\sigma)$, while $\mathrm{des}(\sigma)$ indicates the cardinality of $\mathrm{Des}(\sigma)$.
Observe that the descent set of a permutation $\sigma \in C_n$ must be mirror symmetric, namely, $i \in \mathrm{Des}(\sigma)$ whenever $n - i \in \mathrm{Des}(\sigma)$.

2.2. **Lattice paths.** A *Dyck prefix* is a lattice path in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of up-steps $U = (1, 1)$ and down steps $D = (1, -1)$, and never passing below the x-axis. It is well known (see e.g. [4]) that the number of Dyck prefixes of length n is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. A Dyck prefix ending at ground level is a *Dyck path*. If it is not the case, it will be called a *proper* Dyck prefix.

A *return* of a Dyck prefix is a down step ending on the x-axis. Needless to say, a Dyck prefix is a Dyck path whenever it has a return at the last position. We say that a Dyck prefix is *elevated* if either it has no return, or it has only one return at the last position.

We observe that a given a Dyck prefix $\mathscr{D}$ can be classified according to the position of its last return (*last return decomposition*). The path $\mathscr{D}$ can be:

- a Dyck path
- an elevated proper Dyck prefix
- the juxtaposition of a Dyck path and an elevated proper prefix.

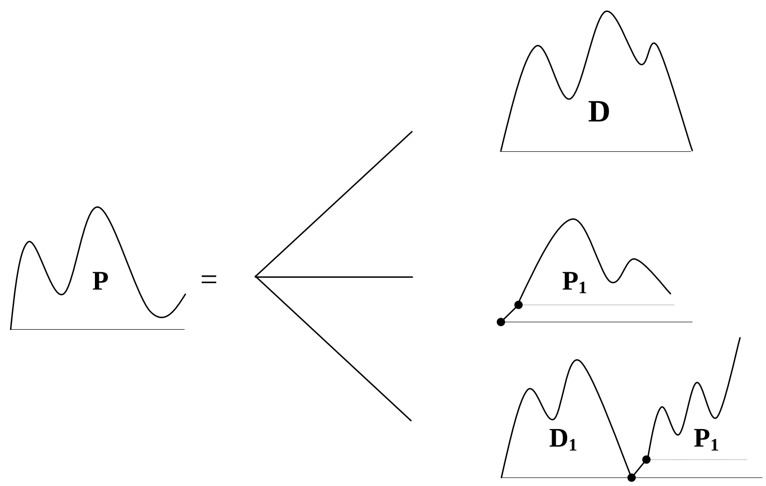

FIGURE 1. The last return decomposition of a Dyck prefix.

3. The descent distribution over the set $C_n(132)$

We begin with two straightforward considerations about centrosymmetric permutations avoiding 132.

- a permutation σ belongs to $C_n(132)$ if and only if the sequence $\sigma(1)\ldots\sigma(n)$ is either $1\,2\ldots n$, or a sequence of the following kind
$$y\,y+1\,\ldots\,n\,\beta\,1\,2\,\ldots\,n+1-y$$
where $y > \left\lceil \frac{n}{2} \right\rceil$ and β, after renormalization, is either empty or a permutation in $C_{2y-2-n}(132)$. For example, the eight permutations in $C_6(132)$ are

123456		456123	($\beta = \emptyset$)
563412	($\beta = 12$)	564312	($\beta = 21$)
623451	($\beta = 1234$)	645231	($\beta = 3412$)
653421	($\beta = 4231$)	654321	($\beta = 4321$)

- the set $C_{2n}(132)$ corresponds bijectively to the set $C_{2n+1}(132)$. In fact, every permutation $\sigma \in C_{2n}(132)$ corresponds to the permutation $\alpha \in C_{2n+1}(132)$ defined as follows:

$$\alpha(i) = \begin{cases} \sigma(i) & \text{if } i \leq n, \\ n+1 & \text{if } i = n+1 \\ \sigma(i-1) & \text{if } i > n+1 \end{cases}$$

For example, $C_7(132)$ contains the following eight permutations:

1234567	5674123
6734512	6754312
7234561	7564231
7634521	7654321

Denote by $q_{n,k}$ (respectively $r_{n,k}$) the number of elements in $C_{2n}(132)$ (resp. $C_{2n+1}(132)$) with k descents, and by

$$Q(x,y) = \sum_{n\geq 0} \sum_{\sigma \in C_{2n}(132)} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d\geq 0} q_{n,d} x^n y^d,$$

$$R(x,y) = \sum_{n\geq 0} \sum_{\sigma \in C_{2n+1}(132)} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d\geq 0} r_{n,d} x^n y^d,$$

the generating functions of the two sequences.

Consider the even case. First of all, $q_{0,0} = 1$ and $q_{n,0} = q_{n,1} = 1$ for every $n > 0$. Moreover, the above characterization for the elements in $C_{2n}(132)$ yields the following recurrence for $q_{n,k}$, with $k \geq 2$:

$$q_{n,k} = \sum_{i=1}^{n-1} q_{i,k-2}.$$

These considerations imply immediately the following:

Theorem 1. *We have:*

$$Q(x,y) = \frac{x(1+y)}{1-x(1+y^2)}.$$

Hence, for every $n \geq 1$,

$$q_{n,k} = \binom{n-1}{\lfloor \frac{k}{2} \rfloor}.$$

◇

Now we turn to the odd case. A permutation $\alpha \in C_{2n+1}(132)$ corresponds to a unique permutation $\sigma \in C_{2n}(132)$. Observe that, if σ has an odd number of descents, then one of these descents is placed at the middle position, and hence α has an additional descent. In the other

case, σ and α have the same number of descents. These considerations imply that $r_{n,0} = 1$ and

$$r_{n,k} = \begin{cases} q_{n,k} + q_{n,k-1} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

for every $k \geq 1$. This yields the following:

Theorem 2. *We have:*

$$R(x, y) = \frac{x}{1 - x(1 + y^2)}.$$

Hence, for every $n \geq 1$,

$$r_{n,k} = \begin{cases} \binom{n}{\frac{k}{2}} & \textit{if } k \textit{ is even} \\ 0 & \textit{if } k \textit{ is odd} \end{cases}$$

◇

4. Characterization of the set $C_n(123)$

The characterization of centrosymmetric 123-avoiding permutations on an odd number of objects is quite simple. In fact, we recall that every permutation $\sigma \in C_{2n+1}$ has a fixed point at $n+1$. Hence, $\sigma \in C_{2n+1}$ avoids 123 whenever it has the following structure:

$$\sigma = \alpha' \ n{+}1 \ \alpha,$$

where α is an arbitrary 123-avoiding permutation on $\{1, 2, \ldots, n\}$ and α' is the sequence of the complements to $2n{+}2$ of the integers $\alpha(n) \cdots \alpha(1)$. For instance, if $\alpha = 7\,6\,4\,3\,2\,1\,5$, we have $\sigma = 11\,15\,14\,13\,12\,10\,9\,8\,7\,6\,4\,3\,2\,1\,5$.

Denote by $v_{n,k}$ the number of permutations in $C_{2n+1}(123)$ with k descents and by

$$V(x, y) = \sum_{n \geq 0} \sum_{\sigma \in C_{2n+1}(123)} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d \geq 0} v_{n,d} x^n y^d,$$

the bivariate generating function of the sequence $v_{n,k}$.

Proposition 3. *The series $V(x, y)$ has the following explicit expression:*

$$V(x, y) = \frac{-1 + \sqrt{1 - 4xy^2 - 4x^2y^2 + 4x^2y^4}}{2xy^2(-1 - x + xy^2)} \tag{1}$$

Proof. Previous arguments show that the integer $v_{2n+1,2k+2}$ equals the number of permutations in $S_n(123)$ with exactly k descents. Hence, we have:

$$V(x, y) = 1 + y^2(E(x, y) - 1),$$

where $E(x,y)$ is the generating function of the Eulerian numbers over $S_n(123)$. It is shown in [2] that

$$E(x,y) = \frac{-1 + 2xy + 2x^2y - 2xy^2 - 4x^2y^2 + 2x^2y^3 + \sqrt{1 - 4xy - 4x^2y + 4x^2y^2}}{2xy^2(xy - 1 - x)}.$$

Trivial computations lead to Identity (1). $\diamond$

We turn now to the even case, and characterize the elements of $C_{2n}(123)$ by means of the well known decomposition of a permutation according to its left-to-right minima (recall that a permutation σ has a *left-to-right minimum* at position i if $\sigma(i) \leq \sigma(j)$ for every $j \leq i$).

First of all, we observe that a centrosymmetric permutation $\sigma \in C_{2n}$ is completely determined by its first n values, namely, by the word

$$w(\sigma) = \sigma(1)\, \sigma(2)\, \ldots\, \sigma(n),$$

and that $w(\sigma)$ can be written as:

$$w(\sigma) = x_1\, w_1\, x_2\, w_2\, \ldots x_k\, w_k,$$

where the integers x_i are the left-to-right minima of σ appearing within the first n positions and w_j are (possibly empty) words. Denote by l_i the length of the word w_i.

In order to characterize the elements of $C_{2n}(123)$, we define a family of alphabets $A_0, A_1, \ldots$ as follows:

- $A_0 = \{1, 2, \ldots, 2n\}$,
- A_i, with $i > 0$, is obtained from A_{i-1} by removing:
  - the integer x_i and its complement $2n + 1 - x_i$, and
  - the integers appearing in w_i together with the corresponding complements.

For every set $A_i = \{s_1, s_2, \ldots, s_{2h_i}\}$, $s_1 < s_2 < \cdots < s_{2h_i}$, we single out its *middle element* $m(A_i) = s_{h_i}$.

We have now immediately the following characterization of the permutations in $C_{2n}(123)$:

Proposition 4. *A centrosymmetric permutation σ avoids* 123 *if and only if*

$$w(\sigma) = x_1\, w_1\, w(\sigma'),$$

where

- $x_1 \geq n$,
- $w_1 = 2n\, 2n-1\, \ldots\, 2n - l_1 + 1$, *with* $2n - l_1 + 1 > x_1$,
- σ' *is a centrosymmetric* 123*-avoiding permutation over the alphabet* A_1,

- *the first entry in σ' is less than x_1.*

$\diamond$

As a consequence, we have:

Corollary 5. *Let σ be a permutation in $C_{2n}(123)$, with*

$$w(\sigma) = x_1\, w_1\, x_2\, w_2\, \ldots x_s\, w_s. \tag{2}$$

Then, for every $i \geq 1$,

$$x_i \geq m(A_{i-1}) \tag{3}$$

where $m(A_{i-1})$ is the middle element of the alphabet A_{i-1}.

If equality holds in (3), x_i will be called a *tiny* minimum. It is easily checked that, if x_i is a tiny minimum, the left-to-right minimum x_j is also tiny, for every $j > i$.

For example, consider the permutation $\sigma = 11\ 16\ 15\ 9\ 7\ 14\ 13\ 12\ 5\ 4\ 3\ 10\ 8\ 2\ 1\ 6$ in $C_{16}(123)$. Then:

$$w(\sigma) = \underbrace{11}_{x_1}\ \underbrace{16\ 15}_{w_1}\ \underbrace{9}_{x_2}\ \underbrace{7}_{x_3}\ \underbrace{14\ 13\ 12}_{w_3}$$

In this case, w_2 is empty, and σ' is the 123-avoiding permutation, order isomorphic to $6\ 4\ 10\ 9\ 8\ 3\ 2\ 1\ 7\ 5$, over the alphabet $A_1 = \{3, 4, 5, 7, 8, 9, 10, 12, 13, 14\}$. Note that 7 is the only tiny minimum in σ.

5. A bijection with Dyck prefixes

We recursively define a map $\Phi : C(123) \to \mathscr{P}$, where $C(123)$ is the set of centrosymmetric 123-avoiding permutations of any finite even length and $\mathscr{P}$ is the set of finite Dyck prefixes of even length. This map associates a permutation $\sigma \in C_{2n}(123)$ with a Dyck prefix of length $2n$ as follows: decompose $w(\sigma)$ as in Identity 2. The word $x_2\, w_2\, \ldots\, x_s\, w_s$, after renormalization, is the word $w(\sigma')$ of some permutation $\sigma' \in C_{2n-2l_1-2}$, where l_i is the length of the word w_i. Now set $k = 2n+1-x_1$. Then:

- if $k < n + 1$, then

  $$\Phi(\sigma) = U^k D^{l_1+1} \bar{\Phi}(\sigma'),$$

  where $\bar{\Phi}(\sigma')$ is the Dyck prefix obtained from $\Phi(\sigma')$ by deleting the leftmost $k - l_1 - 1$ steps;

- if $k = n+1$, namely, x_1 is tiny, then
$$\Phi(\sigma) = U^{k+1} D^{l_1} \hat{\Phi}(\sigma''),$$
where $\hat{\Phi}(\sigma')$ is the Dyck prefix is obtained from $\Phi(\sigma')$ by deleting the leftmost $k - l_1 - 2$ steps.

It is easy to check that the word $\Phi(\sigma)$ is a Dyck prefix.

For example, consider the permutation $\sigma = 11\ 16\ 15\ 9\ 7\ 14\ 13\ 12\ 5\ 4\ 3\ 10\ 8\ 2\ 1\ 6$. Then, $\Phi(\sigma) = U^6 D^3 U^2 DUD^3$ (see Figure 2).

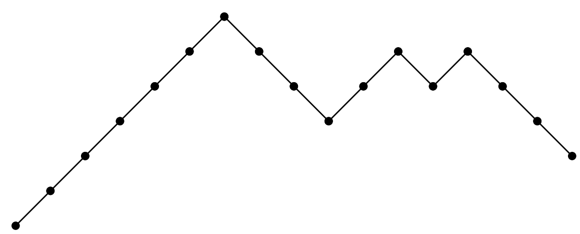

FIGURE 2. The Dyck prefix $\Phi(11\ 16\ 15\ 9\ 7\ 14\ 13\ 12\ 5\ 4\ 3\ 10\ 8\ 2\ 1\ 6)$.

In Figure 3, the prefixes associated with the six permutations in $C_4(123)$ are shown.

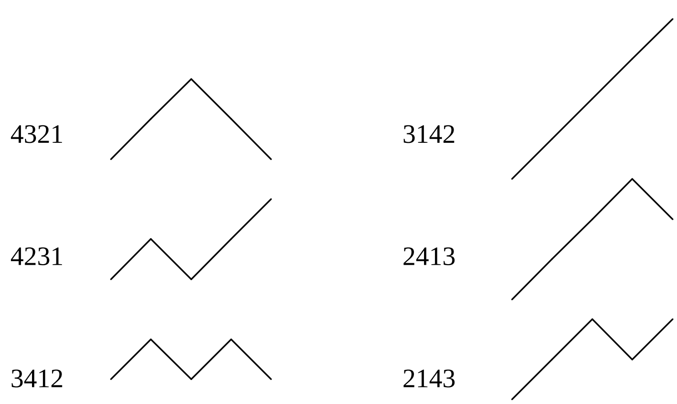

FIGURE 3. The Dyck prefixes $\Phi(\sigma)$, with $\sigma \in C_4(123)$.

The map Φ is a bijection for every positive integer n. In fact, the inverse map $\Phi^{-1} : \mathscr{P} \to C(123)$ can be recursively defined. Consider a Dyck prefix $\pi = U^j D^k \pi'$ of length $2n$, where π' is a (possibly empty) lattice path. The permutation $\sigma = \Phi^{-1}(\pi)$ is defined as follows:

- if $j \leq n$, set

$$\sigma(1) = 2n{+}1{-}j, \quad \sigma(2) = 2n, \quad \sigma(3) = 2n{-}1, \quad \cdots, \quad \sigma(k) = 2n{-}k{+}2$$
$$\sigma(2n) = j, \quad \sigma(2n{-}1) = 1, \quad \sigma(2n{-}2) = 2, \quad \cdots, \quad \sigma(2n{+}1{-}k) = k{-}1,$$

  and let the word $\sigma(k+1)\dots\sigma(2n-k)$ be the permutation of the set $[2n] \setminus \{1, 2, \dots, k-1, j, 2n+1-j, 2n-k+2, \dots, 2n\}$ that is order isomorphic to $\Phi^{-1}(U^{j-k}\pi')$;
- if $j = n+1$, set

$$\sigma(1) = n, \quad \sigma(2) = 2n, \quad \sigma(3) = 2n-1, \quad \cdots, \quad \sigma(k+1) = 2n-k+1$$
$$\sigma(2n) = n{+}1, \quad \sigma(2n{-}1) = 1, \quad \sigma(2n{-}2) = 2, \quad \cdots, \quad \sigma(2n{-}k) = k,$$

  and let the word $\sigma(k+1)\dots\sigma(2n-k)$ be the permutation of the set $[2n] \setminus \{1, 2, \dots, k, n, n+1, 2n-k+1, \dots, 2n\}$ that is order isomorphic to $\Phi^{-1}(U^{j-k-2}\pi')$;
- if $j > n{+}1$, set $\sigma(1) = n$, $\sigma(2n) = n{+}1$, and let $\sigma(2)\dots\sigma(2n{-}1)$ be the permutation of the set $[2n] \setminus \{n, n+1\}$ that is order isomorphic to $\Phi^{-1}(U^{j-2}D^k\pi')$.

For example, the permutation associated with the Dyck prefix $U^3D^2U^6D^2U^2D$ in Figure 4 is $\sigma = 14\ 16\ 8\ 15\ 13\ 7\ 6\ 12\ 5\ 11\ 10\ 4\ 2\ 9\ 1\ 3$.

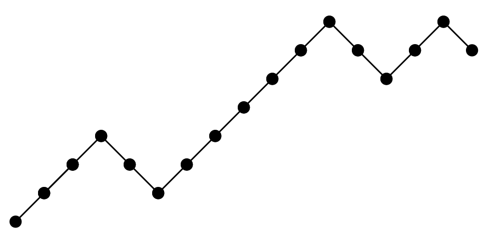

FIGURE 4. The Dyck prefix $U^3D^2U^6D^2U^2D$.

As an immediate consequence, we obtain the following result, previously stated in [3]:

Proposition 6. *The cardinality of the set $C_{2n}(123)$ is the central binomial coefficient $\binom{2n}{n}$.* ◇

6. Properties of the bijection Φ

Some of the properties of a permutation in $C_{2n}(123)$ are related to suitable properties of the associated Dyck prefix.
First of all, the number of tiny minima of a permutation σ determines the height of the last point of its image under Φ. The following result is an immediate consequence of the definition of the map Φ:

Theorem 7. *Let σ be a permutation in $C_{2n}(123)$. The y-coordinate of the last point of the path $\Phi(\sigma)$ is twice the number of tiny minima in σ.*

◇

In particular, we can characterize the permutations corresponding to Dyck paths as follows:

Corollary 8. *Let σ be a permutation in $C_{2n}(123)$. The path $\Phi(\sigma)$ is a Dyck path if and only if σ has no tiny minimum.*

◇

Observe that the permutation σ has no tiny minimum if and only if the word $w(\sigma)$ is a permutation of the set $\{n+1, \ldots, 2n\}$. It is easy to verify that the restriction of Φ to this set of permutations is a slightly modified version of the map described by Krattenthaler in [6].

Consider a permutation $\sigma \in C_{2n}(123)$ with no tiny minima. The bijection Φ is based on a procedure that associates a Dyck prefix with σ, processing the word $w(\sigma)$ from left to right. This allows us to determine at each step of the procedure the height of the last point of the lattice path constructed hitherto. More precisely, we will denote by P_i the Dyck prefix obtained after processing x_i and by Q_i the Dyck prefix obtained after processing w_i. We are interested in determining the heights $k(P_i)$ and $k(Q_i)$ of the last point in P_i and Q_i, respectively. By the definition of Φ, we have $k(P_1) = 2n - x_1$ and $k(Q_1) = 2n - x_1 - l_1$. Consider now the prefix P_2. The alphabet A_1 consists of $2n - 2 - 2l_1$ symbols and x_2 is the $(x_2 - 1 - l_1)$-th smallest element in A_1. Hence, $k(P_2) = (2n - 2 - 2l_1) - (x_2 - 1 - l_1) = 2n - 1 - x_2 - l_1$ and $k(Q_2) = 2n - 1 - x_2 - l_1 - l_2$. Note that these values do not depend on x_1. By similar arguments, we get the following:

$$k(P_j) = 2n - (j-1) - x_j - \sum_{r=1}^{j-1} l_r,$$

$$k(Q_j) = 2n - (j-1) - x_j - \sum_{r=1}^{j} l_r.$$

The previous considerations allow us to relate the right connected components of permutation σ to the returns of the prefix $\Phi(\sigma)$. More precisely, we have:

Theorem 9. *For every $n > 0$, the number of right connected components of $\sigma \in C_{2n}(123)$ is*

$$\begin{array}{ll} 2 \cdot \mathrm{ret}(\Phi(\sigma)) & \text{if } \Phi(\sigma) \text{ is a Dyck path} \\ 2 \cdot \mathrm{ret}(\Phi(\sigma)) + 1 & \text{otherwise} \end{array}$$

where $\mathrm{ret}(\Phi(\sigma))$ *is the number of returns of* $\Phi(\sigma)$.

Proof Let $\hat{D}$ be the first return of $\Phi(\sigma)$, if it exists. Then, if we remove all the steps in $\Phi(\sigma)$ placed after $\hat{D}$, we obtain a Dyck path $\hat{\mathscr{D}}$. Such a Dyck path corresponds to a subword $w' = x_1\, w_1\, \ldots\, x_t\, w_t$ of $w(\sigma)$. By previous remarks, the integers x_i are non-tiny minima. Recall that the last point of $\hat{\mathscr{D}}$ has height

$$k(Q_t) = 2n - (t-1) - x_t - \sum_{r=1}^{t} l_r.$$

The path $\hat{\mathscr{D}}$ is a Dyck path whenever $k(Q_t) = 0$, and this is equivalent to

$$x_t = 2n - (t-1) - \sum_{r=1}^{t} l_r,$$

which is also equivalent to the fact that the set of the entries in w' is the interval $[2n+1-z, 2n]$, with

$$z = t + \sum_{r=1}^{t} l_r.$$

Denote by $w(\sigma) = a_1 \ldots a_n$. Then, the subwords $w' = a_1 \ldots a_z$ and $w'' = 2n+1-a_z \ldots 2n+1-a_1$ are connected components of the permutation σ.
Then, we remove from σ the two subwords w' and w'', and we obtain a new permutation $\tilde{\sigma}$. We repeat this process $\mathrm{ret}(\Phi(\sigma))$ times, ending with a Dyck prefix that is either empty or with no returns. In the first case, the number of connected components of σ is $2{\cdot}\mathrm{ret}(\Phi(\sigma))$. The above considerations imply that in the second case we get a further connected component. $\diamond$

7. The Eulerian distribution on $C_{2n}(123)$

We now study the distribution of the descent statistic over the set $C_{2n}(123)$. To this aim, we consider the bivariate generating function

$$T(x,y) = \sum_{n \geq 0} \sum_{\sigma \in C_{2n}(123)} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d \geq 0} t_{n,d} x^n y^d,$$

where $t_{n,d}$ is the number of permutations in $C_{2n}(123)$ with d descents.

Recall that the descent set of a permutation $\sigma \in C_{2n}(123)$ must be mirror symmetric. This implies that:

$$\mathrm{des}(\sigma) = \begin{cases} 2 \cdot \mathrm{des}(w(\sigma)) & \text{if } \sigma(n) \leq n \\ 2 \cdot \mathrm{des}(w(\sigma)) + 1 & \text{otherwise.} \end{cases}$$

The bijection Φ described and studied in the previous sections reveals to be an effective tool in the analysis of the Eulerian distribution on the set $C_{2n}(123)$. In fact, it is possible to formulate the condition that σ has a descent at a given position in terms of the associated Dyck path.

We begin with the case of permutations corresponding to those Dyck prefixes that are the elementary blocks in the last return decomposition. More precisely:

- the set K_{2n} of the permutations in $C_{2n}(123)$ such that $\Phi(\sigma)$ is a Dyck path. In this case, we denote by $k_{n,d}$ the corresponding Eulerian number and by $N(x,y)$ the bivariate generating function
$$N(x,y) = \sum_n \sum_{\sigma \in K_{2n}} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d \geq 0} k_{n,d} x^n y^d,$$
- the set CK_{2n} of the permutations in $C_{2n}(123)$ such that $\Phi(\sigma)$ is an elevated Dyck path. We denote by $ck_{n,d}$ the corresponding Eulerian number and by $CN(x,y)$ the bivariate generating function
$$CN(x,y) = \sum_n \sum_{\sigma \in CK_{2n}} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d \geq 0} ck_{n,d} x^n y^d,$$
- the set G_{2n} of the permutations in $C_{2n}(123)$ such that $\Phi(\sigma)$ is an proper elevated Dyck prefix. We denote by $g_{n,d}$ the corresponding Eulerian number and by $S(x,y)$ the bivariate generating function
$$S(x,y) = \sum_n \sum_{\sigma \in G_{2n}} x^n y^{\mathrm{des}(\sigma)} = \sum_{n,d \geq 0} g_{n,d} x^n y^d.$$

First of all we study the relations between the two generating functions $N(x,y)$ and $CN(x,y)$. Note that every permutation $\sigma \in K_{2n}$ has a descent at position n. Moreover:

Proposition 10. *Let σ be a permutation in K_{2n}. The number of descents of σ is*
$$des(\sigma) = 2(k_1 + k_2) + 1,$$
*where k_1 is the number of occurrences of DDD (*triple falls*) in $\Phi(\sigma)$ and k_2 is the number of valleys of $\Phi(\sigma)$.*

Proof Let $w(\sigma) = x_1 w_1 \ldots x_k w_k$. A descent in $w(\sigma)$ may occur in one of the two following positions:

1. between two consecutive symbols a and b in same word w_i. These two symbols correspond to two consecutive down steps in $\Phi(\sigma)$, that are necessarily preceded by a previous down step. In fact, if a is not the first symbol in w_i, then a is preceded by a symbol c, that also corresponds to a down step. On the other hand, if a is preceded by x_i in $w(\sigma)$, then x_i corresponds to the collection of steps $U^k D$, since x_i can not be tiny, as remarked in the previous section;
2. before every left-to-right minimum x_i, except for the first one. These positions correspond exactly to the valleys of $\Phi(\sigma)$.

This implies that $\mathrm{des}(w(\sigma)) = k_1 + k_2$. The assertion now follows from the previous considerations. ◇

An elevated Dyck path of length $2n$ with p valleys and q triple falls can be obtained by prepending U and appending D to a Dyck path of length $2n-2$ of one of the two following types:

1. a Dyck path with p valleys and q triple falls, ending with UD,
2. a Dyck path with p valleys and $q-1$ triple falls, not ending with UD.

We note that:

1. the paths of the first kind are in bijection with Dyck paths of length $2n-4$ with $p-1$ valleys and q triple falls;
2. in order to enumerate the paths of the second kind we have to subtract from the number of Dyck paths of length $2n-2$ with p valleys and $q-1$ triple falls the number of Dyck paths of semilength $n-1$ with p valleys and $q-1$ triple falls, ending with UD. Dyck paths of this kind are in bijection with Dyck paths of length $2n-4$ with $p-1$ valleys and $q-1$ triple falls.

Hence, we have:

$$ck_{n,d} = k_{n-1,d-2} - k_{n-2,d-4} + k_{n-2,d-2} \quad (n \geq 2). \tag{4}$$

In addition, exploiting the last return decomposition of a Dyck path, we obtain the following identity, that is a straightforward consequence of Proposition 10:

$$k_{n,d} = ck_{n,d} + \sum_{i=1}^{n-1} \sum_{j=1}^{d-2} ck_{i,j} k_{n-i,d-1-j} \quad (n \geq 3), \tag{5}$$

with the convention $k_{n,d} = 0 = ck_{n,d} = 0$ if $d < 0$.

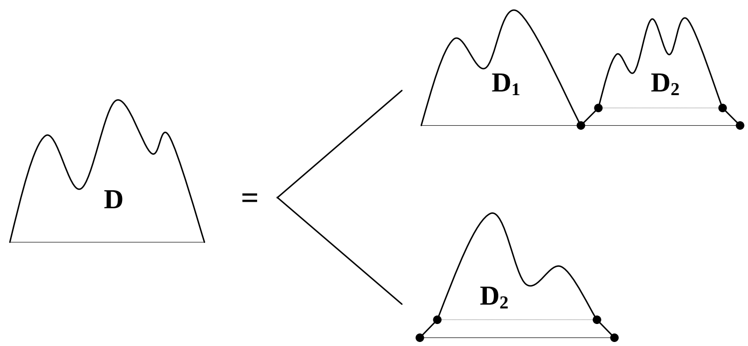

FIGURE 5. The last return decomposition of a Dyck path.

In fact, as remarked in the previous section, if a Dyck prefix $\mathscr{D} = \Phi(\sigma)$ is the juxtaposition of a Dyck path $\mathscr{D}'$ and an elevated proper Dyck prefix $\mathscr{D}''$, then $w(\sigma) = w'\,w''$, where w' contains the greatest symbols in $[1, 2n]$. Hence, $\text{des}(\sigma) = \text{des}(\Phi^{-1}(\mathscr{D}')) + \text{des}(\Phi^{-1}(\mathscr{D}'')) + 1$.
Identities (4) and (5) yield:

$$CK(x,y) = xy^2(K(x,y)-1-xy)+x^2y^2(1-y^2)(K(x,y)-1)+1+xy+x^2y,$$

$$K(x,y) = CK(x,y) + y(CK(x,y)-1)(K(x,y)-1).$$

We deduce the following:

$$xy^3(1-xy^2+x)(K(x,y)-1)^2+(2xy^2+2x^2y^2-2x^2y^4-1)(K(x,y)-1)$$
$$+xy(1-xy^2+x) = 0$$

and hence
(6)
$$K(x,y) = 1+\frac{1-2xy^2-2x^2y^2+2x^2y^4-\sqrt{1-4xy^2-4x^2y^2+4x^2y^4}}{2xy^3(1-xy^2+x)}$$

This completes the case of permutations corresponding to Dyck paths.

Now we turn to the general case. We decompose an arbitrary Dyck prefix according to its last return, getting

Proposition 11. *For every $n \geq 2$, we have*

$$t_{n,d} = g_{n,d} + k_{n,d} + \sum_{i=1}^{n-1}\sum_{j\geq 0} g_{i,j}k_{n-i,d-1-j}. \tag{7}$$

Proof If σ is neither in G_{2n} nor in K_{2n}, then the Dyck prefix $\Phi(\sigma)$ is the juxtaposition of a Dyck path $\mathscr{D}'$ and an elevated proper Dyck prefix $\mathscr{D}''$. In this case, σ can be decomposed as:

$$\sigma = \tau_1 \, \tau_2 \, \tau_3,$$

where the word τ_2, after renormalization, is the permutation $\alpha = \Phi^{-1}(\mathscr{D}'')$ while $\tau_1 \, \tau_3$, after renormalization, is the permutation $\beta = \Phi^{-1}(\mathscr{D}')$. This implies that des$(\sigma)$ =des(α)+des$(\beta) + 2$. $\diamond$

Finally, we express the series $S(x, y)$ in terms of the functions $T(x, y)$ and $K(x, y)$. Note that, given a Dyck prefix $\mathscr{D}$ of length $2n - 2$, we can prepend to $\mathscr{D}$ an up step and append either an up or a down step, hence obtaining two elevated Dyck prefixes $\mathscr{D}'$ and $\mathscr{D}''$.

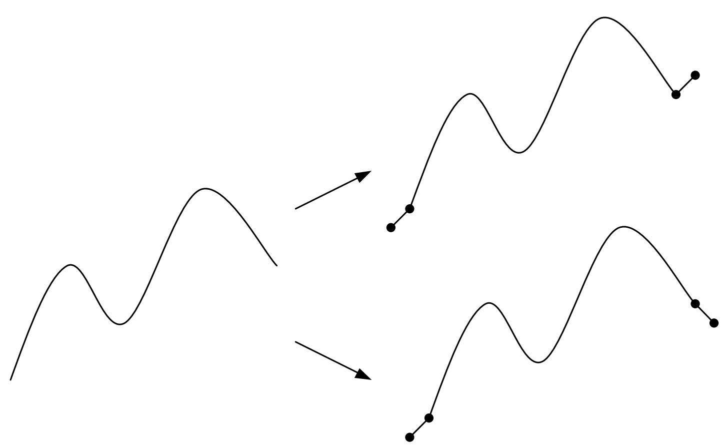

FIGURE 6. The generation of two new Dyck prefixes of length $2n$ from a Dyck prefix of length $2n - 2$.

The prefix $\mathscr{D}'$ is always proper, while $\mathscr{D}''$ is proper whenever the prefix $\mathscr{D}$ is not a Dyck path.
Denote by σ the permutation $\Phi^{-1}(\mathscr{D})$ and suppose that σ has d descents. We want to show that, if we set $\sigma' = \Phi^{-1}(\mathscr{D}')$ and $\sigma''\Phi^{-1}(\mathscr{D}'')$, we have:

$$\{\text{des}(\sigma'), \text{des}(\sigma'')\} = \{d + 1, d + 2\}.$$

Note that the number of descents of the permutations σ' and σ'' depends on the last step in $\mathscr{D}$:

- if the last step of $\mathscr{D}$ is an up step, the last entry of the word $w(\sigma)$ is a tiny minimum. Hence, the word $w(\sigma')$ ends with two consecutive tiny minima, and des(σ') =des(σ)+2. On the other

hand, $w(\sigma'')$ ends with a word w_k of length 1. Hence, σ'' has $d+1$ descents;

- if the last step of $\mathscr{D}$ is a down step, in this case, the descent at position n in σ splits into 2 descents of σ'. Hence, $\mathrm{des}(\sigma') = \mathrm{des}(\sigma) + 1$. Moreover, neither the last entry of the word $w(\sigma)$ nor the last entry of the word $w(\sigma'')$ is a left-to-right minimum. Hence, $\mathrm{des}(\sigma'') = \mathrm{des}(\sigma) + 2$.

Then, we have:

$$g_{n,d} = t_{n-1,d-1} + t_{n-1,d-2} - k_{n-1,d-2} \quad (n \geq 2). \tag{8}$$

with the convention $g_{n,d} = 0$ and $t_{n,d} = 0$ if $d < 0$. Identities (7) and (8) yield the relations:

$$T(x,y) = K(x,y) + S(x,y) - 1 + y(K(x,y) - 1)(S(x,y) - 1),$$

$$S(x,y) = 1 + x + xy(T(x,y) - 1) + xy^2 T(x,y) - xy^2 K(x,y).$$

We deduce the following expression of $T(x,y)$ in terms of $K(x,y)$:

Proposition 12. *We have*

$$T(x,y) = \frac{-xy^3K^2(x,y) + (1 - 2xy^2 + xy + xy^3)K(x,y) + xy^2 - 2xy + x}{1 - xy + xy^3 - xy^2K(x,y) - xy^3K(x,y)}. \tag{9}$$

◇

An explicit expression for the series $T(x,y)$ can be obtained by combining Identities (6) and (9).

The first values of the sequence $t_{n,d}$ are shown in the following table:

n/d	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	0	2	3	1						
3	0	0	3	9	7	1				
4	0	0	0	6	20	28	15	1		
5	0	0	0	0	10	50	85	75	31	1

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